# 6.3 SPECIAL FUNCTIONS

**REVIEW MATERIAL**

- Sections 6.1 and 6.2

**INTRODUCTION** The two differential equations

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1}$$

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{2}$$

occur in advanced studies in applied mathematics, physics, and engineering. They are called **Bessel's equation of order** $\boldsymbol{\nu}$ and **Legendre's equation of order** $\boldsymbol{n}$, respectively. When we solve (1) we shall assume that $\nu \geq 0$, whereas in (2) we shall consider only the case when $n$ is a nonnegative integer.

## 6.3.1 BESSEL'S EQUATION

**THE SOLUTION** Because $x = 0$ is a regular singular point of Bessel's equation, we know that there exists at least one solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting the last expression into (1) gives

$$\begin{aligned} x^2y'' + xy' + (x^2 - \nu^2)y &= \sum_{n=0}^{\infty} c_n(n + r)(n + r - 1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n + r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= c_0(r^2 - r + r - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n + r)(n + r - 1) + (n + r) - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= c_0(r^2 - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n + r)^2 - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}. \end{aligned} \tag{3}$$

From (3) we see that the indicial equation is $r^2 - \nu^2 = 0$, so the indicial roots are $r_1 = \nu$ and $r_2 = -\nu$. When $r_1 = \nu$, (3) becomes

$$\begin{aligned} x^\nu \sum_{n=1}^{\infty} c_n n(n + 2\nu)x^n &+ x^\nu \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= x^\nu \left[(1 + 2\nu)c_1 x + \underbrace{\sum_{n=2}^{\infty} c_n n(n + 2\nu)x^n}_{k = n - 2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{k = n}\right] \\ &= x^\nu \left[(1 + 2\nu)c_1 x + \sum_{k=0}^{\infty} [(k + 2)(k + 2 + 2\nu)c_{k+2} + c_k]x^{k+2}\right] = 0. \end{aligned}$$

Therefore by the usual argument we can write $(1 + 2\nu)c_1 = 0$ and

$$(k + 2)(k + 2 + 2\nu)c_{k+2} + c_k = 0$$

or

$$c_{k+2} = \frac{-c_k}{(k + 2)(k + 2 + 2\nu)}, \qquad k = 0, 1, 2, \ldots. \tag{4}$$

The choice $c_1 = 0$ in (4) implies that $c_3 = c_5 = c_7 = \cdots = 0$, so for $k = 0, 2, 4, \ldots$ we find, after letting $k + 2 = 2n$, $n = 1, 2, 3, \ldots$, that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n + \nu)}. \tag{5}$$

Thus
$$\begin{aligned}
c_2 &= -\frac{c_0}{2^2 \cdot 1 \cdot (1+\nu)} \\
c_4 &= -\frac{c_2}{2^2 \cdot 2(2+\nu)} = \frac{c_0}{2^4 \cdot 1 \cdot 2(1+\nu)(2+\nu)} \\
c_6 &= -\frac{c_4}{2^2 \cdot 3(3+\nu)} = -\frac{c_0}{2^6 \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)} \\
&\ \vdots \\
c_{2n} &= \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \qquad n = 1, 2, 3, \ldots. \qquad (6)
\end{aligned}$$

It is standard practice to choose $c_0$ to be a specific value, namely,

$$c_0 = \frac{1}{2^\nu \Gamma(1+\nu)},$$

where $\Gamma(1+\nu)$ is the gamma function. See Appendix I. Since this latter function possesses the convenient property $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, we can reduce the indicated product in the denominator of (6) to one term. For example,

$$\Gamma(1+\nu+1) = (1+\nu)\Gamma(1+\nu)$$
$$\Gamma(1+\nu+2) = (2+\nu)\Gamma(2+\nu) = (2+\nu)(1+\nu)\Gamma(1+\nu).$$

Hence we can write (6) as

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!(1+\nu)(2+\nu)\cdots(n+\nu)\Gamma(1+\nu)} = \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)}$$

for $n = 0, 1, 2, \ldots$.

**BESSEL FUNCTIONS OF THE FIRST KIND** Using the coefficients $c_{2n}$ just obtained and $r = \nu$, a series solution of (1) is $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$. This solution is usually denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}. \qquad (7)$$

If $\nu \geq 0$, the series converges at least on the interval $[0, \infty)$. Also, for the second exponent $r_2 = -\nu$ we obtain, in exactly the same manner,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}. \qquad (8)$$

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order $\nu$ and $-\nu$, respectively. Depending on the value of $\nu$, (8) may contain negative powers of $x$ and hence converges on $(0, \infty)$.*

Now some care must be taken in writing the general solution of (1). When $\nu = 0$, it is apparent that (7) and (8) are the same. If $\nu > 0$ and $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not a positive integer, it follows from Case I of Section 6.2 that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of (1) on $(0, \infty)$, and so the general solution on the interval is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$. But we also know from Case II of Section 6.2 that when $r_1 - r_2 = 2\nu$ is a positive integer, a second series solution of (1) *may* exist. In this second case we distinguish two possibilities. When $\nu = m =$ positive integer, $J_{-m}(x)$ defined by (8) and $J_m(x)$ are not linearly independent solutions. It can be shown that $J_{-m}$ is a constant multiple of $J_m$ (see Property (*i*) on page 245). In addition, $r_1 - r_2 = 2\nu$ can be a positive integer when $\nu$ is half an odd positive integer. It can be shown in this

*When we replace $x$ by $|x|$, the series given in (7) and (8) converge for $0 < |x| < \infty$.

latter event that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In other words, the general solution of (1) on $(0, \infty)$ is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \qquad \nu \neq \text{integer}. \tag{9}$$

The graphs of $y = J_0(x)$ and $y = J_1(x)$ are given in Figure 6.3.1.

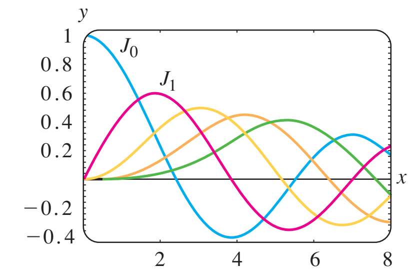


**FIGURE 6.3.1** Bessel functions of the first kind for $n = 0, 1, 2, 3, 4$

## EXAMPLE 1 Bessel's Equation of Order $\frac{1}{2}$

By identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$, we can see from (9) that the general solution of the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ on $(0, \infty)$ is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$. ■

**BESSEL FUNCTIONS OF THE SECOND KIND** If $\nu \neq$ integer, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{10}$$

and the function $J_\nu(x)$ are linearly independent solutions of (1). Thus another form of the general solution of (1) is $y = c_1J_\nu(x) + c_2Y_\nu(x)$, provided that $\nu \neq$ integer. As $\nu \to m$, $m$ an integer, (10) has the indeterminate form $0/0$. However, it can be shown by L'Hôpital's Rule that $\lim_{\nu\to m} Y_\nu(x)$ exists. Moreover, the function

$$Y_m(x) = \lim_{\nu\to m} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of $x^2y'' + xy' + (x^2 - m^2)y = 0$. Hence for *any* value of $\nu$ the general solution of (1) on $(0, \infty)$ can be written as

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{11}$$

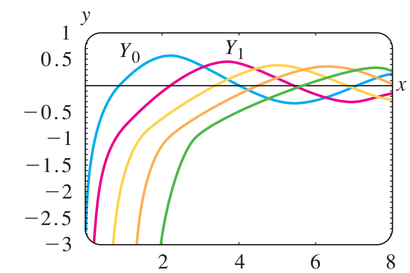


**FIGURE 6.3.2** Bessel functions of the second kind for $n = 0, 1, 2, 3, 4$

$Y_\nu(x)$ is called the **Bessel function of the second kind** of order $\nu$. Figure 6.3.2 shows the graphs of $Y_0(x)$ and $Y_1(x)$.

## EXAMPLE 2 Bessel's Equation of Order 3

By identifying $\nu^2 = 9$ and $\nu = 3$, we see from (11) that the general solution of the equation $x^2y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$ is $y = c_1J_3(x) + c_2Y_3(x)$. ■

**DEs SOLVABLE IN TERMS OF BESSEL FUNCTIONS** Sometimes it is possible to transform a differential equation into equation (1) by means of a change of variable. We can then express the solution of the original equation in terms of Bessel functions. For example, if we let $t = \alpha x$, $\alpha > 0$, in

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \tag{12}$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha\frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

Accordingly, (12) becomes

$$\left(\frac{t}{\alpha}\right)^2\alpha^2\frac{d^2y}{dt^2} + \left(\frac{t}{\alpha}\right)\alpha\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \qquad \text{or} \qquad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

The last equation is Bessel's equation of order $\nu$ with solution $y = c_1J_\nu(t) + c_2Y_\nu(t)$. By resubstituting $t = \alpha x$ in the last expression, we find that the general solution of (12) is

$$y = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x). \tag{13}$$

Equation (12), called the **parametric Bessel equation of order $\nu$,** and its general solution (13) are very important in the study of certain boundary-value problems involving partial differential equations that are expressed in cylindrical coordinates.

Another equation that bears a resemblance to (1) is the **modified Bessel equation of order $\nu$,**

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{14}$$

This DE can be solved in the manner just illustrated for (12). This time if we let $t = ix$, where $i^2 = -1$, then (14) becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

Because solutions of the last DE are $J_\nu(t)$ and $Y_\nu(t)$, *complex-valued* solutions of (14) are $J_\nu(ix)$ and $Y_\nu(ix)$. A real-valued solution, called the **modified Bessel function of the first kind** of order $\nu$, is defined in terms of $J_\nu(ix)$:

$$I_\nu(x) = i^{-\nu}J_\nu(ix). \tag{15}$$

See Problem 21 in Exercises 6.3. Analogous to (10), the **modified Bessel function of the second kind** of order $\nu \neq$ integer is defined to be

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}, \tag{16}$$

and for integer $\nu = n$,

$$K_n(x) = \lim_{\nu \to n} K_\nu(x).$$

Because $I_\nu$ and $K_\nu$ are linearly independent on the interval $(0, \infty)$ for any value of $\nu$, the general solution of (14) is

$$y = c_1I_\nu(x) + c_2K_\nu(x). \tag{17}$$

Yet another equation, important because many DEs fit into its form by appropriate choices of the parameters, is

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)y = 0, \qquad p \geq 0. \tag{18}$$

Although we shall not supply the details, the general solution of (18),

$$y = x^a\left[c_1J_p(bx^c) + c_2Y_p(bx^c)\right], \tag{19}$$

can be found by means of a change in both the independent and the dependent variables: $z = bx^c$, $y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$. If $p$ is not an integer, then $Y_p$ in (19) can be replaced by $J_{-p}$.

## EXAMPLE 3 Using (18)

Find the general solution of $xy'' + 3y' + 9y = 0$ on $(0, \infty)$.

**SOLUTION** By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0,$$

we can make the following identifications with (18):

$$1 - 2a = 3, \qquad b^2c^2 = 9, \qquad 2c - 2 = -1, \qquad \text{and} \qquad a^2 - p^2c^2 = 0.$$

The first and third equations imply that $a = -1$ and $c = \frac{1}{2}$. With these values the second and fourth equations are satisfied by taking $b = 6$ and $p = 2$. From (19) we find that the general solution of the given DE on the interval $(0, \infty)$ is $y = x^{-1}[c_1 J_2(6x^{1/2}) + c_2 Y_2(6x^{1/2})]$. ■

## EXAMPLE 4 The Aging Spring Revisited

Recall that in Section 5.1 we saw that one mathematical model for the free undamped motion of a mass on an aging spring is given by $mx'' + ke^{-\alpha t}x = 0$, $\alpha > 0$. We are now in a position to find the general solution of the equation. It is left as a problem to show that the change of variables $s = \dfrac{2}{\alpha}\sqrt{\dfrac{k}{m}}\,e^{-\alpha t/2}$ transforms the differential equation of the aging spring into

$$s^2 \frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2 x = 0.$$

The last equation is recognized as (1) with $\nu = 0$ and where the symbols $x$ and $s$ play the roles of $y$ and $x$, respectively. The general solution of the new equation is $x = c_1 J_0(s) + c_2 Y_0(s)$. If we resubstitute $s$, then the general solution of $mx'' + ke^{-\alpha t}x = 0$ is seen to be

$$x(t) = c_1 J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + c_2 Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right).$$

See Problems 33 and 39 in Exercises 6.3. ■

The other model that was discussed in Section 5.1 of a spring whose characteristics change with time was $mx'' + ktx = 0$. By dividing through by $m$, we see that the equation $x'' + \dfrac{k}{m}tx = 0$ is Airy's equation $y'' + \alpha^2 xy = 0$. See Example 3 in Section 6.1. The general solution of Airy's differential equation can also be written in terms of Bessel functions. See Problems 34, 35, and 40 in Exercises 6.3.

**PROPERTIES** We list below a few of the more useful properties of Bessel functions of order $m$, $m = 0, 1, 2, \ldots$:

$$(i)\ \ J_{-m}(x) = (-1)^m J_m(x), \qquad (ii)\ \ J_m(-x) = (-1)^m J_m(x),$$

$$(iii)\ \ J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0, \end{cases} \qquad (iv)\ \ \lim_{x\to 0^+} Y_m(x) = -\infty.$$

Note that Property *(ii)* indicates that $J_m(x)$ is an even function if $m$ is an even integer and an odd function if $m$ is an odd integer. The graphs of $Y_0(x)$ and $Y_1(x)$ in Figure 6.3.2 illustrate Property *(iv)*, namely, $Y_m(x)$ is unbounded at the origin. This last fact is not obvious from (10). The solutions of the Bessel equation of order 0 can be obtained by using the solutions $y_1(x)$ in (21) and $y_2(x)$ in (22) of Section 6.2. It can be shown that (21) of Section 6.2 is $y_1(x) = J_0(x)$, whereas (22) of that section is

$$y_2(x) = J_0(x)\ln x - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}.$$

The Bessel function of the second kind of order 0, $Y_0(x)$, is then defined to be the linear combination $Y_0(x) = \dfrac{2}{\pi}(\gamma - \ln 2)y_1(x) + \dfrac{2}{\pi}y_2(x)$ for $x > 0$. That is,

$$Y_0(x) = \frac{2}{\pi}J_0(x)\left[\gamma + \ln\frac{x}{2}\right] - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k},$$

where $\gamma = 0.57721566\ldots$ is **Euler's constant.** Because of the presence of the logarithmic term, it is apparent that $Y_0(x)$ is discontinuous at $x = 0$.

**NUMERICAL VALUES** The first five nonnegative zeros of $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ are given in Table 6.1. Some additional function values of these four functions are given in Table 6.2.

**TABLE 6.1** Zeros of $J_0$, $J_1$, $Y_0$, and $Y_1$

| $J_0(x)$ | $J_1(x)$ | $Y_0(x)$ | $Y_1(x)$ |
|---|---|---|---|
| 2.4048 | 0.0000 | 0.8936 | 2.1971 |
| 5.5201 | 3.8317 | 3.9577 | 5.4297 |
| 8.6537 | 7.0156 | 7.0861 | 8.5960 |
| 11.7915 | 10.1735 | 10.2223 | 11.7492 |
| 14.9309 | 13.3237 | 13.3611 | 14.8974 |

**TABLE 6.2** Numerical Values of $J_0$, $J_1$, $Y_0$, and $Y_1$

| $x$ | $J_0(x)$ | $J_1(x)$ | $Y_0(x)$ | $Y_1(x)$ |
|---|---|---|---|---|
| 0 | 1.0000 | 0.0000 | — | — |
| 1 | 0.7652 | 0.4401 | 0.0883 | −0.7812 |
| 2 | 0.2239 | 0.5767 | 0.5104 | −0.1070 |
| 3 | −0.2601 | 0.3391 | 0.3769 | 0.3247 |
| 4 | −0.3971 | −0.0660 | −0.0169 | 0.3979 |
| 5 | −0.1776 | −0.3276 | −0.3085 | 0.1479 |
| 6 | 0.1506 | −0.2767 | −0.2882 | −0.1750 |
| 7 | 0.3001 | −0.0047 | −0.0259 | −0.3027 |
| 8 | 0.1717 | 0.2346 | 0.2235 | −0.1581 |
| 9 | −0.0903 | 0.2453 | 0.2499 | 0.1043 |
| 10 | −0.2459 | 0.0435 | 0.0557 | 0.2490 |
| 11 | −0.1712 | −0.1768 | −0.1688 | 0.1637 |
| 12 | 0.0477 | −0.2234 | −0.2252 | −0.0571 |
| 13 | 0.2069 | −0.0703 | −0.0782 | −0.2101 |
| 14 | 0.1711 | 0.1334 | 0.1272 | −0.1666 |
| 15 | −0.0142 | 0.2051 | 0.2055 | 0.0211 |

**DIFFERENTIAL RECURRENCE RELATION** Recurrence formulas that relate Bessel functions of different orders are important in theory and in applications. In the next example we derive a **differential recurrence relation.**

## EXAMPLE 5 Derivation Using the Series Definition

Derive the formula $xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$.

**SOLUTION** It follows from (7) that

$$\begin{aligned} xJ'_\nu(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} \\ &= \nu\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} + 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} \\ &= \nu J_\nu(x) + x\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}}_{k=n-1} \\ &= \nu J_\nu(x) - x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu+1} = \nu J_\nu(x) - xJ_{\nu+1}(x). \end{aligned}$$

∎

The result in Example 5 can be written in an alternative form. Dividing $xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$ by $x$ gives

$$J'_\nu(x) - \frac{\nu}{x}J_\nu(x) = -J_{\nu+1}(x).$$

This last expression is recognized as a linear first-order differential equation in $J_\nu(x)$. Multiplying both sides of the equality by the integrating factor $x^{-\nu}$ then yields

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x). \tag{20}$$

It can be shown in a similar manner that

$$\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu}J_{\nu-1}(x). \tag{21}$$

See Problem 27 in Exercises 6.3. The differential recurrence relations (20) and (21) are also valid for the Bessel function of the second kind $Y_\nu(x)$. Observe that when $\nu = 0$, it follows from (20) that

$$J_0'(x) = -J_1(x) \qquad \text{and} \qquad Y_0'(x) = -Y_1(x). \tag{22}$$

An application of these results is given in Problem 39 of Exercises 6.3.

**SPHERICAL BESSEL FUNCTIONS** When the order $\nu$ is half an odd integer, that is, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, the Bessel functions of the first kind $J_\nu(x)$ can be expressed in terms of the elementary functions $\sin x$, $\cos x$, and powers of $x$. Such Bessel functions are called **spherical Bessel functions.** Let's consider the case when $\nu = \frac{1}{2}$. From (7),

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(1 + \frac{1}{2} + n\right)} \left(\frac{x}{2}\right)^{2n+1/2}.$$

In view of the property $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ and the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ the values of $\Gamma\left(1 + \frac{1}{2} + n\right)$ for $n = 0$, $n = 1$, $n = 2$, and $n = 3$ are, respectively,

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1 + \tfrac{1}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(1 + \tfrac{3}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(1 + \tfrac{5}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \frac{5 \cdot 3}{2^3}\sqrt{\pi} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^3 4 \cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{9}{2}\right) = \Gamma\left(1 + \tfrac{7}{2}\right) = \tfrac{7}{2}\Gamma\left(\tfrac{7}{2}\right) = \frac{7 \cdot 5}{2^6 \cdot 2!}\sqrt{\pi} = \frac{7 \cdot 6 \cdot 5!}{2^6 \cdot 6 \cdot 2!}\sqrt{\pi} = \frac{7!}{2^7 3!}\sqrt{\pi}.$$

In general,
$$\Gamma\left(1 + \frac{1}{2} + n\right) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}.$$

Hence
$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\dfrac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Since the infinite series in the last line is the Maclaurin series for $\sin x$, we have shown that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{23}$$

It is left as an exercise to show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{24}$$

See Problems 31 and 32 in Exercises 6.3.

## 6.3.2 LEGENDRE'S EQUATION

**THE SOLUTION** Since $x = 0$ is an ordinary point of Legendre's equation (2), we substitute the series $y = \sum_{k=0}^{\infty} c_k x^k$, shift summation indices, and combine series to get

$$(1 - x^2)y'' - 2xy' + n(n+1)y = [n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x + \sum_{j=2}^{\infty} [(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j]x^j = 0$$

which implies that

$$\begin{aligned} n(n+1)c_0 + 2c_2 &= 0 \\ (n-1)(n+2)c_1 + 6c_3 &= 0 \\ (j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j &= 0 \end{aligned}$$

or

$$\begin{aligned} c_2 &= -\frac{n(n+1)}{2!}c_0 \\ c_3 &= -\frac{(n-1)(n+2)}{3!}c_1 \\ c_{j+2} &= -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, \qquad j = 2, 3, 4, \ldots . \end{aligned} \tag{25}$$

If we let $j$ take on the values 2, 3, 4, . . . , the recurrence relation (25) yields

$$\begin{aligned} c_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0 \\ c_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1 \\ c_6 &= -\frac{(n-4)(n+5)}{6 \cdot 5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0 \\ c_7 &= -\frac{(n-5)(n+6)}{7 \cdot 6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1 \end{aligned}$$

and so on. Thus for at least $|x| < 1$ we obtain two linearly independent power series solutions:

$$\begin{aligned} y_1(x) &= c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 \right. \\ &\qquad \left. - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right] \\ y_2(x) &= c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \right. \\ &\qquad \left. - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]. \end{aligned} \tag{26}$$

Notice that if $n$ is an even integer, the first series terminates, whereas $y_2(x)$ is an infinite series. For example, if $n = 4$, then

$$y_1(x) = c_0\left[1 - \frac{4 \cdot 5}{2!}x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!}x^4\right] = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right].$$

Similarly, when $n$ is an odd integer, the series for $y_2(x)$ terminates with $x^n$; that is, *when $n$ is a nonnegative integer, we obtain an $n$th-degree polynomial solution* of Legendre's equation.

Because we know that a constant multiple of a solution of Legendre's equation is also a solution, it is traditional to choose specific values for $c_0$ or $c_1$, depending on whether $n$ is an even or odd positive integer, respectively. For $n = 0$ we choose $c_0 = 1$, and for $n = 2, 4, 6, \ldots$

$$c_0 = (-1)^{n/2}\frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n},$$

whereas for $n = 1$ we choose $c_1 = 1$, and for $n = 3, 5, 7, \ldots$

$$c_1 = (-1)^{(n-1)/2}\frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}.$$

For example, when $n = 4$, we have

$$y_1(x) = (-1)^{4/2}\frac{1 \cdot 3}{2 \cdot 4}\left[1 - 10x^2 + \frac{35}{3}x^4\right] = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

**LEGENDRE POLYNOMIALS** These specific $n$th-degree polynomial solutions are called **Legendre polynomials** and are denoted by $P_n(x)$. From the series for $y_1(x)$ and $y_2(x)$ and from the above choices of $c_0$ and $c_1$ we find that the first several Legendre polynomials are

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x, \\ &P_2(x) = \frac{1}{2}(3x^2 - 1), && P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned} \tag{27}$$

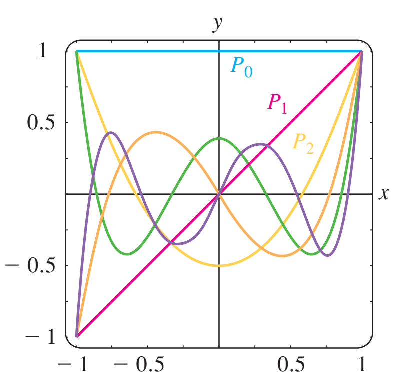


**FIGURE 6.3.3** Legendre polynomials for $n = 0, 1, 2, 3, 4, 5$

Remember, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, . . . are, in turn, particular solutions of the differential equations

$$\begin{aligned} n = 0:&\quad (1 - x^2)y'' - 2xy' = 0, \\ n = 1:&\quad (1 - x^2)y'' - 2xy' + 2y = 0, \\ n = 2:&\quad (1 - x^2)y'' - 2xy' + 6y = 0, \\ n = 3:&\quad (1 - x^2)y'' - 2xy' + 12y = 0, \\ \vdots&\qquad\qquad \vdots \end{aligned} \tag{28}$$

The graphs, on the interval $[-1, 1]$, of the six Legendre polynomials in (27) are given in Figure 6.3.3.

**PROPERTIES** You are encouraged to verify the following properties using the Legendre polynomials in (27).

$$(i)\ \ P_n(-x) = (-1)^n P_n(x)$$

$$(ii)\ \ P_n(1) = 1 \qquad\qquad (iii)\ \ P_n(-1) = (-1)^n$$

$$(iv)\ \ P_n(0) = 0, \quad n \text{ odd} \qquad\qquad (v)\ \ P_n'(0) = 0, \quad n \text{ even}$$

Property ($i$) indicates, as is apparent in Figure 6.3.3, that $P_n(x)$ is an even or odd function according to whether $n$ is even or odd.

**RECURRENCE RELATION** Recurrence relations that relate Legendre polynomials of different degrees are also important in some aspects of their applications. We state, without proof, the three-term recurrence relation

$$(k + 1)P_{k+1}(x) - (2k + 1)xP_k(x) + kP_{k-1}(x) = 0, \tag{29}$$

which is valid for $k = 1, 2, 3, \ldots$. In (27) we listed the first six Legendre polynomials. If, say, we wish to find $P_6(x)$, we can use (29) with $k = 5$. This relation expresses $P_6(x)$ in terms of the known $P_4(x)$ and $P_5(x)$. See Problem 45 in Exercises 6.3.

Another formula, although not a recurrence relation, can generate the Legendre polynomials by differentiation. **Rodrigues' formula** for these polynomials is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n, \qquad n = 0, 1, 2, \ldots. \tag{30}$$

See Problem 48 in Exercises 6.3.

## REMARKS

(*i*) Although we have assumed that the parameter $n$ in Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, represented a nonnegative integer, in a more general setting $n$ can represent any real number. Any solution of Legendre's equation is called a **Legendre function.** If $n$ is *not* a nonnegative integer, then both Legendre functions $y_1(x)$ and $y_2(x)$ given in (26) are infinite series convergent on the open interval $(-1, 1)$ and divergent (unbounded) at $x = \pm 1$. If $n$ is a nonnegative integer, then as we have just seen one of the Legendre functions in (26) is a polynomial and the other is an infinite series convergent for $-1 < x < 1$. You should be aware of the fact that Legendre's equation possesses solutions that are bounded on the *closed* interval $[-1, 1]$ only in the case when $n = 0, 1, 2, \ldots$. More to the point, the only Legendre functions that are bounded on the closed interval $[-1, 1]$ are the Legendre polynomials $P_n(x)$ or constant multiples of these polynomials. See Problem 47 in Exercises 6.3 and Problem 24 in Chapter 6 in Review.

(*ii*) In the *Remarks* at the end of Section 2.3 we mentioned the branch of mathematics called **special functions.** Perhaps a better appellation for this field of applied mathematics might be *named functions,* since many of the functions studied bear proper names: Bessel functions, Legendre functions, Airy functions, Chebyshev polynomials, Gauss's hypergeometric function, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Mathieu functions, Weber functions, and so on. Historically, special functions were the by-product of necessity; someone needed a solution of a very specialized differential equation that arose from an attempt to solve a physical problem.

# EXERCISES 6.3

*Answers to selected odd-numbered problems begin on page ANS-10.*

## 6.3.1 BESSEL'S EQUATION

In Problems 1–6 use (1) to find the general solution of the given differential equation on $(0, \infty)$.

**1.** $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

**2.** $x^2y'' + xy' + (x^2 - 1)y = 0$

**3.** $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$

**4.** $16x^2y'' + 16xy' + (16x^2 - 1)y = 0$

**5.** $xy'' + y' + xy = 0$

**6.** $\dfrac{d}{dx}[xy'] + \left(x - \dfrac{4}{x}\right)y = 0$

In Problems 7–10 use (12) to find the general solution of the given differential equation on $(0, \infty)$.

**7.** $x^2y'' + xy' + (9x^2 - 4)y = 0$

**8.** $x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

**9.** $x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

**10.** $x^2y'' + xy' + (2x^2 - 64)y = 0$

In Problems 11 and 12 use the indicated change of variable to find the general solution of the given differential equation on $(0, \infty)$.

**11.** $x^2y'' + 2xy' + \alpha^2x^2y = 0; \quad y = x^{-1/2}v(x)$

**12.** $x^2y'' + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)y = 0; \quad y = \sqrt{x}\,v(x)$

In Problems 13–20 use (18) to find the general solution of the given differential equation on $(0, \infty)$.

**13.** $xy'' + 2y' + 4y = 0$ **14.** $xy'' + 3y' + xy = 0$

**15.** $xy'' - y' + xy = 0$ **16.** $xy'' - 5y' + xy = 0$

**17.** $x^2y'' + (x^2 - 2)y = 0$

**18.** $4x^2y'' + (16x^2 + 1)y = 0$

**19.** $xy'' + 3y' + x^3y = 0$

**20.** $9x^2y'' + 9xy' + (x^6 - 36)y = 0$

**21.** Use the series in (7) to verify that $I_\nu(x) = i^{-\nu}J_\nu(ix)$ is a real function.

**22.** Assume that $b$ in equation (18) can be pure imaginary, that is, $b = \beta i, \beta > 0, i^2 = -1$. Use this assumption to express the general solution of the given differential equation in terms the modified Bessel functions $I_n$ and $K_n$.

**(a)** $y'' - x^2y = 0$ **(b)** $xy'' + y' - 7x^3y = 0$

In Problems 23–26 first use (18) to express the general solution of the given differential equation in terms of Bessel functions. Then use (23) and (24) to express the general solution in terms of elementary functions.

**23.** $y'' + y = 0$

**24.** $x^2y'' + 4xy' + (x^2 + 2)y = 0$

**25.** $16x^2y'' + 32xy' + (x^4 - 12)y = 0$

**26.** $4x^2y'' - 4xy' + (16x^2 + 3)y = 0$

**27.** **(a)** Proceed as in Example 5 to show that

$$xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x).$$

[*Hint*: Write $2n + \nu = 2(n + \nu) - \nu$.]

**(b)** Use the result in part (a) to derive (21).

**28.** Use the formula obtained in Example 5 along with part (a) of Problem 27 to derive the recurrence relation

$$2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x).$$

In Problems 29 and 30 use (20) or (21) to obtain the given result.

**29.** $\displaystyle\int_0^x rJ_0(r)\,dr = xJ_1(x)$ **30.** $J'_0(x) = J_{-1}(x) = -J_1(x)$

**31.** Proceed as on page 247 to derive the elementary form of $J_{-1/2}(x)$ given in (24).

**32.** **(a)** Use the recurrence relation in Problem 28 along with (23) and (24) to express $J_{3/2}(x)$, $J_{-3/2}(x)$, and $J_{5/2}(x)$ in terms of $\sin x$, $\cos x$, and powers of $x$.

**(b)** Use a graphing utility to graph $J_{1/2}(x)$, $J_{-1/2}(x)$, $J_{3/2}(x)$, $J_{-3/2}(x)$, and $J_{5/2}(x)$.

**33.** Use the change of variables $s = \dfrac{2}{\alpha}\sqrt{\dfrac{k}{m}}\,e^{-\alpha t/2}$ to show that the differential equation of the aging spring $mx'' + ke^{-\alpha t}x = 0, \alpha > 0$, becomes

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0.$$

**34.** Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2xy = 0, x > 0$, whenever $w$ is a solution of Bessel's equation of order $\frac{1}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0, t > 0$. [*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$.]

**35.** **(a)** Use the result of Problem 34 to express the general solution of Airy's differential equation for $x > 0$ in terms of Bessel functions.

**(b)** Verify the results in part (a) using (18).

**36.** Use the Table 6.1 to find the first three positive eigenvalues and corresponding eigenfunctions of the boundary-value problem

$$xy'' + y' + \lambda xy = 0,$$

$$y(x), y'(x) \text{ bounded as } x \to 0^+, \quad y(2) = 0.$$

[*Hint*: By identifying $\lambda = \alpha^2$, the DE is the parametric Bessel equation of order zero.]

**37.** **(a)** Use (18) to show that the general solution of the differential equation $xy'' + \lambda y = 0$ on the interval $(0, \infty)$ is

$$y = c_1\sqrt{x}J_1\left(2\sqrt{\lambda x}\right) + c_2\sqrt{x}Y_1\left(2\sqrt{\lambda x}\right).$$

**(b)** Verify by direct substitution that $y = \sqrt{x}J_1(2\sqrt{x})$ is a particular solution of the DE in the case $\lambda = 1$.

## Computer Lab Assignments

**38.** Use a CAS to graph the modified Bessel functions $I_0(x)$, $I_1(x)$, $I_2(x)$ and $K_0(x)$, $K_1(x)$, $K_2(x)$. Compare these graphs with those shown in Figures 6.3.1 and 6.3.2. What major difference is apparent between Bessel functions and the modified Bessel functions?

**39.** **(a)** Use the general solution given in Example 4 to solve the IVP

$$4x'' + e^{-0.1t}x = 0, \quad x(0) = 1, \quad x'(0) = -\tfrac{1}{2}.$$

Also use $J'_0(x) = -J_1(x)$ and $Y'_0(x) = -Y_1(x)$ along with Table 6.1 or a CAS to evaluate coefficients.

**(b)** Use a CAS to graph the solution obtained in part (a) for $0 \le t \le \infty$.

**40.** **(a)** Use the general solution obtained in Problem 35 to solve the IVP

$$4x'' + tx = 0, \quad x(0.1) = 1, \quad x'(0.1) = -\tfrac{1}{2}.$$

Use a CAS to evaluate coefficients.

**(b)** Use a CAS to graph the solution obtained in part (a) for $0 \le t \le 200$.

**41.** **Column Bending Under Its Own Weight** A uniform thin column of length $L$, positioned vertically with one end embedded in the ground, will deflect, or bend away, from the vertical under the influence of its own weight when its length or height exceeds a certain critical value. It can be shown that the angular deflection $\theta(x)$ of the column from the vertical at a point $P(x)$ is a solution of the boundary-value problem:

$$EI\frac{d^2\theta}{dx^2} + \delta g(L - x)\theta = 0, \quad \theta(0) = 0, \quad \theta'(L) = 0,$$

where $E$ is Young's modulus, $I$ is the cross-sectional moment of inertia, $\delta$ is the constant linear density, and $x$ is the distance along the column measured from its base. See Figure 6.3.4. The column will bend only for those values of $L$ for which the boundary-value problem has a nontrivial solution.

**(a)** Restate the boundary-value problem by making the change of variables $t = L - x$. Then use the results of a problem earlier in this exercise set to express the general solution of the differential equation in terms of Bessel functions.

**(b)** Use the general solution found in part (a) to find a solution of the BVP and an equation which defines the critical length $L$, that is, the smallest value of $L$ for which the column will start to bend.

**(c)** With the aid of a CAS, find the critical length $L$ of a solid steel rod of radius $r = 0.05$ in., $\delta g = 0.28\,A$ lb/in., $E = 2.6 \times 10^7$ lb/in.$^2$, $A = \pi r^2$, and $I = \frac{1}{4}\pi r^4$.

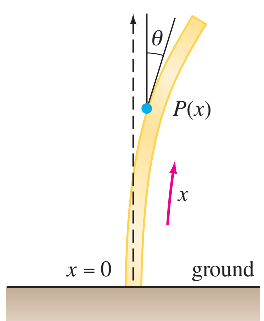


FIGURE 6.3.4 Beam in Problem 41

**42.** **Buckling of a Thin Vertical Column** In Example 3 of Section 5.2 we saw that when a constant vertical compressive force, or load, $P$ was applied to a thin column of uniform cross section and hinged at both ends, the deflection $y(x)$ is a solution of the BVP:

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

**(a)** If the bending stiffness factor $EI$ is proportional to $x$, then $EI(x) = kx$, where $k$ is a constant of proportionality. If $EI(L) = kL = M$ is the maximum stiffness factor, then $k = M/L$ and so $EI(x) = Mx/L$. Use the information in Problem 37 to find a solution of

$$M\frac{x}{L}\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

if it is known that $\sqrt{x}\,Y_1(2\sqrt{\lambda x})$ is *not* zero at $x = 0$.

**(b)** Use Table 6.1 to find the Euler load $P_1$ for the column.

**(c)** Use a CAS to graph the first buckling mode $y_1(x)$ corresponding to the Euler load $P_1$. For simplicity assume that $c_1 = 1$ and $L = 1$.

**43.** **Pendulum of Varying Length** For the simple pendulum described on page 209 of Section 5.3, suppose that the rod holding the mass $m$ at one end is replaced by a flexible wire or string and that the wire is strung over a pulley at the point of support $O$ in Figure 5.3.3. In this manner, while it is in motion in a vertical plane, the mass $m$ can be raised or lowered. In other words, the length $l(t)$ of the pendulum varies with time. Under the same assumptions leading to equation (6) in Section 5.3, it can be shown* that the differential equation for the displacement angle $\theta$ is now

$$l\theta'' + 2l'\theta' + g\sin\theta = 0.$$

**(a)** If $l$ increases at constant rate $v$ and if $l(0) = l_0$, show that a linearization of the foregoing DE is

$$(l_0 + vt)\theta'' + 2v\theta' + g\theta = 0. \tag{31}$$

**(b)** Make the change of variables $x = (l_0 + vt)/v$ and show that (31) becomes

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{g}{vx}\theta = 0.$$

**(c)** Use part (b) and (18) to express the general solution of equation (31) in terms of Bessel functions.

**(d)** Use the general solution obtained in part (c) to solve the initial-value problem consisting of equation (31) and the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$. [*Hints*: To simplify calculations, use a further change of variable $u = \dfrac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\dfrac{g}{v}}\,x^{1/2}$.]

*See *Mathematical Methods in Physical Sciences,* Mary Boas, John Wiley & Sons, Inc., 1966. Also see the article by Borelli, Coleman, and Hobson in *Mathematics Magazine,* vol. 58, no. 2, March 1985.

Also, recall that (20) holds for both $J_1(u)$ and $Y_1(u)$. Finally, the identity

$$J_1(u)Y_2(u) - J_2(u)Y_1(u) = -\frac{2}{\pi u} \text{ will be helpful.]}$$

**(e)** Use a CAS to graph the solution $\theta(t)$ of the IVP in part (d) when $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s. Experiment with the graph using different time intervals such as [0, 10], [0, 30], and so on.

**(f)** What do the graphs indicate about the displacement angle $\theta(t)$ as the length $l$ of the wire increases with time?

### 6.3.2 LEGENDRE'S EQUATION

**44.** **(a)** Use the explicit solutions $y_1(x)$ and $y_2(x)$ of Legendre's equation given in (26) and the appropriate choice of $c_0$ and $c_1$ to find the Legendre polynomials $P_6(x)$ and $P_7(x)$.

**(b)** Write the differential equations for which $P_6(x)$ and $P_7(x)$ are particular solutions.

**45.** Use the recurrence relation (29) and $P_0(x) = 1$, $P_1(x) = x$, to generate the next six Legendre polynomials.

**46.** Show that the differential equation

$$\sin\theta\,\frac{d^2y}{d\theta^2} + \cos\theta\,\frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$

can be transformed into Legendre's equation by means of the substitution $x = \cos\theta$.

**47.** Find the first three positive values of $\lambda$ for which the problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

$$y(0) = 0, \quad y(x), y'(x) \text{ bounded on } [-1,1]$$

has nontrivial solutions.

### Computer Lab Assignments

**48.** For purposes of this problem ignore the list of Legendre polynomials given on page 249 and the graphs given in Figure 6.3.3. Use Rodrigues' formula (30) to generate the Legendre polynomials $P_1(x), P_2(x), \ldots, P_7(x)$. Use a CAS to carry out the differentiations and simplifications.

**49.** Use a CAS to graph $P_1(x), P_2(x), \ldots, P_7(x)$ on the interval $[-1, 1]$.

**50.** Use a root-finding application to find the zeros of $P_1(x), P_2(x), \ldots, P_7(x)$. If the Legendre polynomials are built-in functions of your CAS, find zeros of Legendre polynomials of higher degree. Form a conjecture about the location of the zeros of any Legendre polynomial $P_n(x)$, and then investigate to see whether it is true.

## CHAPTER 6 IN REVIEW

*Answers to selected odd-numbered problems begin on page ANS-10.*

In Problems 1 and 2 answer true or false without referring back to the text.

**1.** The general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$ is $y = c_1J_1(x) + c_2J_{-1}(x)$. ________

**2.** Because $x = 0$ is an irregular singular point of $x^3y'' - xy' + y = 0$, the DE possesses no solution that is analytic at $x = 0$. ________

**3.** Both power series solutions of $y'' + \ln(x + 1)y' + y = 0$ centered at the ordinary point $x = 0$ are guaranteed to converge for all $x$ in which *one* of the following intervals?

**(a)** $(-\infty, \infty)$ **(b)** $(-1, \infty)$

**(c)** $[-\frac{1}{2}, \frac{1}{2}]$ **(d)** $[-1, 1]$

**4.** $x = 0$ is an ordinary point of a certain linear differential equation. After the assumed solution $y = \sum_{n=0}^{\infty} c_nx^n$ is substituted into the DE, the following algebraic system is obtained by equating the coefficients of $x^0$, $x^1$, $x^2$, and $x^3$ to zero:

$$\begin{aligned} 2c_2 + 2c_1 + c_0 &= 0 \\ 6c_3 + 4c_2 + c_1 &= 0 \\ 12c_4 + 6c_3 + c_2 - \tfrac{1}{3}c_1 &= 0 \\ 20c_5 + 8c_4 + c_3 - \tfrac{2}{3}c_2 &= 0. \end{aligned}$$

Bearing in mind that $c_0$ and $c_1$ are arbitrary, write down the first five terms of two power series solutions of the differential equation.

**5.** Suppose the power series $\sum_{k=0}^{\infty} c_k(x - 4)^k$ is known to converge at $-2$ and diverge at 13. Discuss whether the series converges at $-7$, 0, 7, 10, and 11. Possible answers are *does, does not, might.*

**6.** Use the Maclaurin series for $\sin x$ and $\cos x$ along with long division to find the first three nonzero terms of a power series in $x$ for the function $f(x) = \dfrac{\sin x}{\cos x}$.

In Problems 7 and 8 construct a linear second-order differential equation that has the given properties.

**7.** A regular singular point at $x = 1$ and an irregular singular point at $x = 0$

**8.** Regular singular points at $x = 1$ and at $x = -3$

In Problems 9–14 use an appropriate infinite series method about $x = 0$ to find two solutions of the given differential equation.

**9.** $2xy'' + y' + y = 0$

**10.** $y'' - xy' - y = 0$

**11.** $(x - 1)y'' + 3y = 0$

**12.** $y'' - x^2y' + xy = 0$

**13.** $xy'' - (x + 2)y' + 2y = 0$

**14.** $(\cos x)y'' + y = 0$

In Problems 15 and 16 solve the given initial-value problem.

**15.** $y'' + xy' + 2y = 0, \quad y(0) = 3, y'(0) = -2$

**16.** $(x + 2)y'' + 3y = 0, \quad y(0) = 0, y'(0) = 1$

**17.** Without actually solving the differential equation $(1 - 2\sin x)y'' + xy = 0$, find a lower bound for the radius of convergence of power series solutions about the ordinary point $x = 0$.

**18.** Even though $x = 0$ is an ordinary point of the differential equation, explain why it is not a good idea to try to find a solution of the IVP

$$y'' + xy' + y = 0, \quad y(1) = -6, \quad y'(1) = 3$$

of the form $y = \sum_{n=0}^{\infty} c_n x^n$. Using power series, find a better way to solve the problem.

In Problems 19 and 20 investigate whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation. [*Hint*: Recall the Maclaurin series for $\cos x$ and $e^x$.]

**19.** $xy'' + (1 - \cos x)y' + x^2y = 0$

**20.** $(e^x - 1 - x)y'' + xy = 0$

**21.** Note that $x = 0$ is an ordinary point of the differential equation $y'' + x^2y' + 2xy = 5 - 2x + 10x^3$. Use the assumption $y = \sum_{n=0}^{\infty} c_n x^n$ to find the general solution $y = y_c + y_p$ that consists of three power series centered at $x = 0$.

**22.** The first-order differential equation $dy/dx = x^2 + y^2$ cannot be solved in terms of elementary functions. However, a solution can be expressed in terms of Bessel functions.

**(a)** Show that the substitution $y = -\dfrac{1}{u}\dfrac{du}{dx}$ leads to the equation $u'' + x^2u = 0$.

**(b)** Use (18) in Section 6.3 to find the general solution of $u'' + x^2u = 0$.

**(c)** Use (20) and (21) in Section 6.3 in the forms

$$J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

and $$J'_\nu(x) = -\frac{\nu}{x}J_\nu(x) + J_{\nu-1}(x)$$

as an aid to show that a one-parameter family of solutions of $dy/dx = x^2 + y^2$ is given by

$$y = x\,\frac{J_{3/4}\left(\frac{1}{2}x^2\right) - cJ_{-3/4}\left(\frac{1}{2}x^2\right)}{cJ_{1/4}\left(\frac{1}{2}x^2\right) + J_{-1/4}\left(\frac{1}{2}x^2\right)}.$$

**23. (a)** Use (23) and (24) of Section 6.3 to show that

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x.$$

**(b)** Use (15) of Section 6.3 to show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x \quad \text{and} \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x.$$

**(c)** Use part (b) to show that

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}.$$

**24. (a)** From (27) and (28) of Section 6.3 we know that when $n = 0$, Legendre's differential equation $(1 - x^2)y'' - 2xy' = 0$ has the polynomial solution $y = P_0(x) = 1$. Use (5) of Section 4.2 to show that a second Legendre function satisfying the DE for $-1 < x < 1$ is

$$y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

**(b)** We also know from (27) and (28) of Section 6.3 that when $n = 1$, Legendre's differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$ possesses the polynomial solution $y = P_1(x) = x$. Use (5) of Section 4.2 to show that a second Legendre function satisfying the DE for $-1 < x < 1$ is

$$y = \frac{x}{2}\ln\left(\frac{1 + x}{1 - x}\right) - 1.$$

**(c)** Use a graphing utility to graph the logarithmic Legendre functions given in parts (a) and (b).

**25. (a)** Use binomial series to formally show that

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

**(b)** Use the result obtained in part (a) to show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. See Properties (*ii*) and (*iii*) on page 249.